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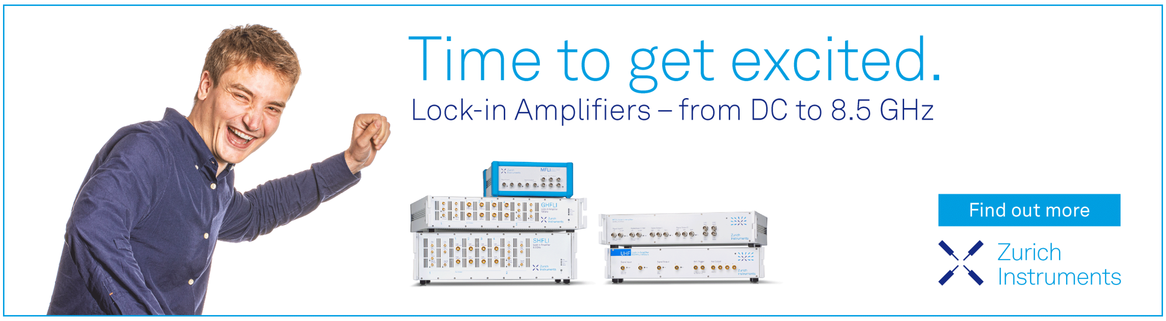

Quadratic Sequences in Pythagorean Triples

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Abstract. Using the Euclid's formula, we obtain an alternative formula for generating Pythagorean triples, both primitive and non-primitive. It easy to classify Pythagorean triples using this formula based on the divisibility of the leg of a Pythagorean triple by any positive integer. The differences in lengths between the hypotenuse and the legs of a Pythagorean triple obtained by this alternative formula form Quadratic sequences. These quadratic sequences have applications in various fields such as tiling.

INTRODUCTION

A Pythagorean Triple (PT) is a triple of positive integers (a, b, c), which satisfy the Pythagorean equation

$$a^2 + b^2 = c^2 \tag{1}$$

where c represents the length of the hypotenuse, a and b represent the lengths of the other two sides (legs) of a Pythagorean triangle. We say a Pythagorean triple (a, b, c) is primitive if the numbers a, b and c are pairwise co-prime[1, 2].

Many methods have been formulated that generate Pythagorean triples, see [3, 4, 5, 6, 7, 8, 9]. The most common one is the classical Greek formula

$$(a,\ b,\ c) = (n^2 - m^2,\ 2nm,\ n^2 + m^2) \tag{2}$$

where $0 < m < n;\ n, m \in \mathbb{Z}^+$. A triple generated by this method is primitive if and only if $(n, m) = 1$ and $(n - m)$ is odd, that is, n, m have opposite parity[1, 2].

From equation (2), the distance between the hypotenuse c; and the leg a; which we denote as $DL(c, a)$ is

$$DL(c, a) = (n^2 + m^2) - (n^2 - m^2) = 2m^2. \tag{3}$$

similarly,

$$DL(c, b) = (n^2 + m^2) - 2nm = (n - m)^2 = u^2, \tag{4}$$

where $u = n - m$.

In [10], it is shown from equations (3) and (4) that

$$(a,\ b,\ c) = (u^2 + 2mu,\ 2mu + 2m^2,\ u^2 + 2mu + 2m^2); \tag{5}$$

is a primitive Pythagorean triple for all positive odd integers u, every $m \in \mathbb{Z}^+$ and $(u, m) = 1$. If u is even for every $m \in \mathbb{Z}^+$, then (5) is non-primitive.

Using the list of Pythagorean triples generated by the difference formula in (5), we formulate sequences that follow from taking the difference between and the sum of $DL(c, a)$ and $DL(c, b)$.

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DIFFERENCE AND SUM OF THE DIFFERENCES BETWEEN THE HYPOTENUSE AND THE LEGS

TABLE 1. Differences between the hypotenuse and the lengths of the legs

u	m	a	b	c	DL(c, a)	DL(c, b)	DDL[(c, a), (c, b)]	SDL[(c, a), (c, b)]
1	1	3	4	5	2	1	1	3
1	2	5	12	13	8	1	7	9
1	3	7	24	25	18	1	17	19
1	4	9	40	41	32	1	31	33
1	5	11	60	61	50	1	49	51
1	6	13	84	85	72	1	71	73
1	7	15	112	113	98	1	97	99
1	8	17	144	145	128	1	127	129
1	9	19	180	181	162	1	161	163
1	10	21	220	221	200	1	199	201
3	1	15	8	17	2	9	11	-7
3	2	21	20	29	8	9	17	-1
3	3	27	36	45	18	9	27	9
3	4	33	56	65	32	9	41	23
3	5	39	80	89	50	9	59	41
3	6	45	108	117	72	9	81	63
3	7	51	140	149	98	9	107	89
3	8	57	176	185	128	9	137	119
3	9	63	216	225	162	9	171	153
3	10	69	260	269	200	9	209	191
5	1	35	12	37	2	25	27	-23
5	2	45	28	53	8	25	33	-17
5	3	55	48	73	18	25	43	-7
5	4	65	72	97	32	25	57	7
5	5	75	100	125	50	25	75	25
5	6	85	132	157	72	25	97	47
5	7	95	168	193	98	25	123	73
5	8	105	208	233	128	25	153	103
5	9	115	252	277	162	25	187	137
5	10	125	300	325	200	25	225	175

We adopt the notation $DDL[(c,\ a),\ (c,b)]$ to be the difference between $DL(c,\ a)$ and $DL(c,\ b)$ and similarly $SDL[(c,\ a),\ (c,b)]$ be the sum of the two.

It is easy to see that:

$$DDL[(c,\ a),\ (c,b)] = 2m^2\ -\ u^2 \tag{6}$$

and

$$SDL[(c,\ a),\ (c,b)] = 2m^2 +\ u^2 \tag{7}$$

Observe that the difference of the differences, $DDL[(c,\ a),\ (c,b)]$, between the hypotenuse and the length of the legs is equal to the distance between the two legs a and b; that is,

$$DDL[(c,a),(c,b)] =\ (c-\ a)-\ (c-\ b) =\ b-\ a\ =\ DL(b,a).$$

Let $u\ = \{1,\ 3,\ 5\}$ and $1\ \leq\ m\ \leq\ 10$, we obtain a list of triples generated from the difference formula, shown in the table 1 abobe. For each of these triples we find the difference between the hypotenuse and each of the legs, $DL(c,a)$ and $DL(c,b)$. We also obtain $DDL[(c,\ a),\ (c,b)]$ and $SDL[(c,\ a),\ (c,b)]$ for various fixed values of u and varying values of m**.** We then formulate sequences that arise from these differences.

Proposition

Let $(a,\ b,\ c)\ =\ (u^2\ +\ 2mu,\ 2mu\ +\ 2m^2,\ u^2\ +\ 2mu\ +\ 2m^2)$ be a Pythagorean triple where u is a positive odd integer and $m \in \mathbb{Z}^+$. The difference of the differences, $\mathrm{DDL}[(c,a),\ (c,b)]$ and sum of the differences $\mathrm{SDL}[(c,a),\ (c,b)]$ between the hypotenuse and the lengths of the legs respectively form the sequences

$$T_n\ =\ 2n^2 -\ u^2 \tag{8}$$

and

$$T_n\ =\ 2n^2 +\ u^2 \tag{9}$$

for a fixed positive odd integer u and $n \in \mathbb{Z}^+$.

PROOF: Suppose $u = 1$; then the sequence of $DDL[(c,a),(c,b)]$ from table 1 is $T_n = 1,\ 7,\ 17,\ 31,\ 49, 71,\ 97,\ \_\_\_$

By inspection, we notice the second differences between each consecutive term differ by the same amount called the common second difference, d. In this case $d\ =\ 4$. This means the sequence is quadratic sequence which has the general form: $T_n\ =\ An^2\ +\ Bn\ +\ C$ where $A, B, C \in \mathbb{Z}^+$ for every $n \in \mathbb{Z}^+$.

We find the second coefficients A, B and C. Use the first three terms of the sequence to set up and solve a system of linear equations. We obtain:

$$T_1\ =\ A(1)^2 +\ B(1)\ +\ C\ =\ A\ +\ B\ +\ C\ =\ 1 \tag{10}$$

$$T_2\ =\ A(2)^2\ +\ B(2)\ +\ C\ =\ 4A\ +\ 2B\ +\ C\ =\ 7 \tag{11}$$

$$T_3\ =\ A(3)^2\ +\ B(3)\ +\ C\ =\ 9A\ +\ 3B\ +\ C\ =\ 17 \tag{12}$$

Subtract (10) from (11) to obtain

$$T_2 - T_1\ =\ 3A\ +\ B\ =\ 6 \tag{13}$$

Similarly, the difference between T_3 and T_2 gives

$$T_3 -\ T_2\ =\ 5A\ +\ B\ =\ 10 \tag{14}$$

Solve (13) and (14) simultaneously to obtain $A\ =\ 2$ and $B\ =\ 0$. Substitute these in (8) to get $C\ =\ -1$. We thus have

$$T_n\ =\ 2n^2 -\ 1$$

It can be shown in a similar way, the sequence of $SDL[(c,a),(c,b)]$, that is,

$$T_n\ =\ 3, 9, 19, 33, 51, 73,\ ---$$

is

$$T_n\ =\ 2n^2 +\ 1$$

Now, since u is odd, we let $u\ =\ 2k\ +1$ for all $k \in \mathbb{Z}^+$. For convenience, we denote u as u_k. We then assume the sequence is true for u_k and show it is true for u_{k+1}.

For the first case, the sequence of the difference of differences, $DDL[(c,a),(c,b)]$ is

$$\mathrm{T_n}\ =\ -4\mathrm{k}^2 -\ 12\mathrm{k} -\ 7,\ -4\mathrm{k}^2 -\ 12\mathrm{k} -\ 1,\ -4\mathrm{k}^2 -\ 12\mathrm{k}\ +\ 9,\ -4\mathrm{k}^2 -\ 12\mathrm{k}\ +\ 23,$$
$$-4k^2 -\ 12k\ +\ 41,\ -4k^2 -\ 12k +\ 63 \ldots$$

We then have

$$T_1\ =\ A\ +\ B\ +\ C\ =\ -4k^2 -\ 12k -\ 7 \tag{15}$$

$$T_2\ =\ 4A\ +\ 2B\ +\ C\ =-\ 4k^2 -\ 12k -\ 1 \tag{16}$$

$$T_3\ =\ 9A\ +\ 3B\ +\ C\ =\ -4k^2 -\ 12k\ +\ 9 \tag{17}$$

Subtract (15) from (16) to obtain

$$T_2 -\ T_1\ =\ 3A\ +\ B\ =\ 6. \tag{18}$$

Similarly, the difference between T_3 and T_2 gives

$$T_3 -\ T_2\ =\ 5A\ +\ B\ =\ 10. \tag{19}$$

Solve (18) and (19) simultaneously to obtain $A\ =\ 2$ and $B\ =\ 0$. Substitute these in (8) to get $C\ =\ -4k^2 -\ 12k -\ 9$. We thus have

$$T_n\ =\ 2n^2 -\ (4k^2\ +\ 12k\ +\ 9)$$
$$\Leftrightarrow T_n\ =\ 2n^2 -\ (2k\ +\ 3)^2$$

$$\Leftrightarrow T_n = 2n^2 - [2(k+1)+1]^2$$

The second sequence of $SDL[(c,a),(c,b)]$, that is,
$T_n = 4k^2 + 12k + 11, 4k^2 + 12k + 17, 4k^2 + 12k + 27, 4k^2 + 12k + 41, 4k^2 + 12k + 59, 4k^2 + 12k + 81, \ldots$
is solved in a similar way to obtain $T_n = 2n^2 + [2(k+1)+1]^2$. Thus by mathematical induction, the sequence is true for all $k \epsilon \mathbb{Z}^+$ and in turn positive odd integer u and $n \epsilon \mathbb{Z}^+$.

SEQUENCES FORMED FOR FIXED VALUES OF m AND VARYING VALUES OF u

When we fix m and vary u, we obtain a series of quadratic sequences. These sequences are formed for both $DDL[(c,a),(c,b)]$ and $SDL[(c,a),(c,b)]$.

Lemma 3.1 The sum of the differences, SDL[(c, a), (c, b)] between the hypotenuse and the length of the legs for each value of m and various values of u form sequences as described in the table 2 below, for all $n \geq 0$.

TABLE 2. Sum of differences between the hypotenuse and the lengths of the legs

m \ *u*	*1*	*3*	*5*	*7*	*9*	...	*sequence*
1	3	*11*	*27*	*51*	*83*	...	$4n^2 - 4n + 3$
2	*9*	*17*	*33*	*57*	*89*	...	$4n^2 - 4n + 9$
3	*19*	*27*	*43*	*67*	*99*	...	$4n^2 - 4n + 19$
4	*33*	*41*	*57*	*81*	*113*	...	$4n^2 - 4n + 33$
5	*51*	*59*	*75*	*99*	*131*	...	$4n^2 - 4n + 51$
6	*73*	*81*	*97*	*121*	*153*	...	$4n^2 - 4n + 73$
7	*99*	*107*	*123*	*147*	*179*	...	$4n^2 - 4n + 99$
8	*129*	*137*	*153*	*177*	*209*	...	$4n^2 - 4n + 129$
9	*163*	*171*	*187*	*211*	*243*	...	$4n^2 - 4n + 163$
10	*201*	*209*	*225*	*249*	*281*	...	$4n^2 - 4n + 201$

PROOF. Let m = 1, we obtain the sequence $T_n = 3, 11, 27, 51, 83, \_\_\_$ Observe the second differences between each consecutive term differ by $d = 8$, thus a quadratic sequence. Then $T_n = An^2 + Bn + C$ where $A, B, C \epsilon \mathbb{Z}^+$ for every $n \epsilon \mathbb{Z}^+$. We determine the values of the coefficients A, B and C from the first three terms of the sequence. We obtain:

$$T_1 = A + B + C = 3 \quad (20)$$

$$T_2 = 4A + 2B + C = 11 \quad (21)$$

$$T_3 = 9A + 3B + C = 27 \quad (22)$$

Subtract (20) from (21) to obtain

$$T_2 - T_1 = 3A + B = 8. \quad (23)$$

Similarly, the difference between T_3 and T_2 gives

$$T_3 - T_2 = 5A + B = 16. \quad (24)$$

Solve (23) and (24) simultaneously to obtain $A = 4$ and $B = -4$. Substitute these in (18) to obtain $C = 3$. We thus have

$$T_n = 4n^2 - 4n + 3.$$

In a similar way, we obtain the sequences for the other values of m.

The next lemma lays out the sequences associated with the difference of the differences, $DDL[(c,a),(c,b)]$ between the hypotenuse and the length of the legs.

Lemma 3.2 The difference of the differences, $DDL[(c,a),(c,b)]$, between the hypotenuse and the length of the legs for each value of m and various values of u, similarly form sequences as described in the table 3 below, for all $n \geq 1$.

TABLE 3. Difference of differences between the hypotenuse and the lengths of the legs

m \ *u*	*1*	*3*	*5*	*7*	*9*	...	*Sequence*
1	1	*-7*	*-23*	*-47*	*-79*	...	$-4n^2+4n+1$
2	*7*	*-1*	*-17*	*-41*	*-73*	...	$-4n^2+4n+7$
3	*17*	*9*	*-7*	*-31*	*-63*	...	$-4n^2+4n+17$
4	*31*	*23*	*7*	*-17*	*-49*	...	$-4n^2+4n+31$
5	*49*	*41*	*25*	*1*	*-31*	...	$-4n^2+4n+49$
6	*71*	*63*	*47*	*23*	*-9*	...	$-4n^2+4n+71$
7	*97*	*89*	*73*	*49*	*17*	...	$-4n^2+4n+97$
8	*127*	*119*	*103*	*79*	*47*	...	$-4n^2+4n+127$
9	*161*	*153*	*137*	*113*	*81*	...	$-4n^2+4n+161$
10	*199*	*191*	*175*	*151*	*119*	...	$-4n^2+4n+199$

PROOF: If m = 1; we have the sequence $T_n = 1, -7, -23, -47, -79, \_\_\_$. This is a quadratic sequence with $d = 8$.

Then $T_n = An^2 + Bn + C$ where $A, B, C \in \mathbb{Z}^+$ for every $n \in \mathbb{Z}^+$. We obtain:

$$T_1 = A + B + C = 1 \tag{25}$$

$$T_2 = 4A + 2B + C = -7 \tag{26}$$

$$T_3 = 9A + 3B + C = -23 \tag{27}$$

Subtract (25) from (26) to obtain

$$T_2 - T_1 = 3A + B = -8: \tag{28}$$

The difference between T_3 and T_2 is

$$T_3 - T_2 = 5A + B = -16. \tag{29}$$

Solve (28) and (29) simultaneously to obtain $A = -4$ and $B = 4$. Substitute these in (25) to get $C = 1$. We thus have

$$T_n = -4n^2 + 4n + 1.$$

The sequences for the other values of m are shown in a similar way.

REMARK 3.1 Observe that the constant terms for varying values of m in tables 2 and 3 form the sequences already shown in proof of proposition 2.1. Also, from table 3, we see that $T_0 = T_1$.

Proposition 2.1 and lemmas 3.1 and 3.2 thus prove the following result.

Proposition

For each fixed positive odd integer u, varying values of $n \in \mathbb{Z}^+$ and fixed integer $m \in \mathbb{Z}^+$ the difference of the differences, $DDL[(c, a), (c, b)]$ between the hypotenuse and the length of the legs form the sequence

$$T_{n,m} = -4n^2 + 4n + (2m^2 - u^2); \qquad n \geq 1 \tag{30}$$

and similarly, the sum of the differences, $SDL[(c, a), (c, b)$ between the hypotenuse and the length of the legs form the sequence

$$T_{n,m} = 4n^2 - 4n + (2m^2 + u^2); \qquad n \geq 1. \tag{31}$$

CONCLUSION

The difference and sum of differences between the hypotenuse and the legs of Pythagorean triples generated by the difference formula form quadratic sequences. These quadratic sequences and in turn the Pythagorean triples from which they are generated have implications in various fields. For instance, these quadratic sequences can be applied in tiling.

For example, to start a spiral of square tiles, trivially the first tile fits in a 1 by 1 square, 7 tiles fit in a 3 by 3 square, 17 tiles fit in a 5 by 5 square and so on [6]. Note that 1, 7, 17, _ _ _ is the sequence of $DDL[(c, a), (c, b)]$ when $u = 1$.

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REFERENCES

1. D. E. Joyce, Euclid's Elements Book X, Proposition XXIX. Clark University, Worcester, (1997).
2. Mitchell, D. W. (July 2001), An Alternative Characterisation of All Primitive Pythagorean Triples, *The Mathematical Gazette*, **85** (503): 27-35.
3. F. R. Bernhart and H. L. Price, Heron's Formula, Descartes circles and Pythagorean Triangles. arXiv:math/0701624, (2005).
4. R. D. Carmichael, The Theory of Numbers and Diophantine Analysis. Dover Publications, New York, (1959).
5. L. E. Dickson, History of the Theory of Numbers, Vol.II: Diophantine Analysis, Carnegie Institution, Washington, DC, (1920).
6. Heino, J. (Dec 13 2009) - https://oeis.org/A056220
7. D. Houston, Pythagorean Triples via double - angle formulas in Nelsen, R. B., Proofs Without Words: Exercises in Visual Thinking. *Mathematical Association of America*, Washington DC, (1993).
8. S. Kak, Pythagorean Triples and Cryptographic Coding. arXiv:1004.3770, (2010).
9. T. Koshy, Elementary Number Theory with Applications, 2nd Edition. Academic Press, San Diego, (2007).
10. Ochieng, R. C., Chikunji, J. C. & 0nyango-Otieno, V., A Note on the Generation of Primitive Pythagorean Triples